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Dynamical Noether symmetries

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Abstract. We use Noether's theorem within a restricted class of dynamical transformations involving velocity dependence to obtain first integral constants of the motion not available as Noether point transformation constants.

1. Introduction

The application of Noether's theorem to Lagrangian systems described by a finite number of coordinates x^i is usually restricted to the class of point transformations

$$t \to \bar{t}(t, x) \qquad x^i \to \bar{x}^i(t, x).$$

Such transformations, however, do not in general exhaust the symmetry of the system, motivating an extension of Noether's results to 'dynamical' transformations which include dependence also on the velocities $v^i \equiv \mathrm{d}x^i/\mathrm{d}t$. A general and modern discussion (within an equivalent Hamiltonian context) is presented by Arnold (1978). However, here we restrict ourselves to the simple case of one-parameter groups of infinitesimal dynamical transformations of the form

$$t \to \bar{t} = t + \varepsilon\xi(t, x, v) \tag{1a}$$

$$x^i \to \bar{x}^i = x^i + \varepsilon\eta^i(t, x, v) \tag{1b}$$

with the descriptors ξ and η^i so restricted as to be at most linear in the velocities. With this added minimal generalisation, we show in the following that we are able to generate extra Noether constants of the motion not obtainable via point symmetries alone.

2. Noether's theorem and its extension to dynamical symmetries

We consider physical systems described by n classical coordinates x^i, $i = 1, 2, \ldots, n$, and satisfying a dynamical law of the form

$$\mathrm{D}v^i/\mathrm{d}t = F^i(x, t) \tag{2}$$

with $v^i \equiv \mathrm{d}x^i/\mathrm{d}t$. The n quantities x^i are taken as coordinates of a point in an n-dimensional manifold endowed with a metric $g_{ij}(x)$ relative to which the absolute derivative $\mathrm{D}/\mathrm{d}t$ is defined in the usual way via Christoffel symbols $\{^{\,i}_{jk}\}$. The time t is

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considered here as a scalar parameter and is allowed to enter into the generalised force vector field F^i as such. Thus from the point of view of (time independent) coordinate transformations on the manifold

$$x^i \to x^{i'} = x^i(x) \tag{3}$$

the equation of motion (2) is to be viewed as a vector equation. In this form it covers the usual equations of mechanics if we take the metric $g_{ij}(x)$ as defined by the kinetic energy

$$T \equiv \tfrac{1}{2} g_{ij} v^i v^j. \tag{4}$$

In this work we shall also restrict to mechanical systems that possess a Lagrangian

$$L \equiv T + V(t, x) \tag{5}$$

as a consequence of which the force vector F^i reduces to a gradient

$$F^i = -g^{ij} V_{,j} \tag{6}$$

with g^{ij} the contravariant metric tensor. A comma denotes both here and in what follows partial differentiation with respect to any subsequent indices. In mechanical situations, one is generally concerned only with a flat manifold where the metric components g_{ij} may, in appropriate coordinates, always be reduced to constants. However, this could be otherwise in some problems, such as particle motion in general relativity in which case the space–time metric is identified with g_{ij} and t plays the role of an invariant path parameter such as proper time.

For subsequent use we expand the absolute derivative in (2) and write the equation of motion also in the alternative non-covariant form

$$\dot{v}^i = -g^{ij} V_{,j} - \{^{\,i}_{j,k}\} v^j v^k \tag{7}$$

with

$$\dot{v}^i \equiv \mathrm{d}v^i/\mathrm{d}t.$$

We now consider general one-parameter groups of infinitesimal dynamical transformations of the coordinates x^i and the time t, as set out in (1), with ε the infinitesimal group parameter and ξ and η^i a set of $n+1$ functions to be determined. This transformation (1) has the 'active' geometrical interpretation of mapping one trajectory into another. In contrast, the (passive) coordinate transformations of (3) do not interchange trajectories; they clearly require the n quantities η^i to transform as a contravariant vector and the single quantity ξ as a scalar (since both t and $\bar{t}$ are to be invariant parameters).

The transformation (1) automatically determines how the successive time derivatives v^i, $\dot{v}^i$, etc, behave under the group. Thus correct to first order in the infinitesimal group parameter ε, we readily deduce that an arbitrary function $\Psi(t, x, v, \ldots)$ experiences a first-order (in ε) change given by

$$\delta\Psi = \varepsilon U \Psi$$

where U is Lie's 'extended' operator

$$U \equiv \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial x^i} + (\dot{\eta}^i - \dot{\xi} v^i) \frac{\partial}{\partial v^i} + \ldots. \tag{8}$$

Here we may truncate the definition (8) to only the first three terms shown, since we shall not require the action of U on functions Ψ involving higher time derivatives than $\dot{v}^i$.

The Noether theorem (Noether 1917, Hill 1951, Lutzky 1979) now asserts the basic result that under the general class of transformations of the type (1), the quantity

$$\Phi \equiv (\xi v^i - \eta^i)(\partial L/\partial v^i) - \xi L + f \tag{9}$$

is a constant of the motion, provided the unknown quantities ξ, η^i and the scalar function $f = f(t, x)$ satisfy the condition

$$UL + \dot{\xi} L - \dot{f} = 0. \tag{10}$$

This criterion may alternatively be expressed in the covariant form

$$g_{ij}(\mathrm{D}\eta^i/\mathrm{d}t)v^j - \eta^i V_{,i} - \xi V_{,t} + \dot{\xi} L - \dot{f} = 0. \tag{10'}$$

Here a comma followed by a subscript t denotes the operation $\partial/\partial t$ and clearly does not affect the tensorial character of the object on which it acts (e.g. $V_{,t}$ is a scalar).

In the conventional applications of Noether's theorem where one restricts to point transformations by setting

$$\xi = \xi(t, x) \qquad \eta^i = \eta^i(t, x)$$

it is possible to decompose the Noether condition (10) generally into a further set of equations (viewed as constraint eliminating conditions) by equating to zero the coefficients of successive powers of the velocity v^i. A solution of these equations then supplies the unknown quantities ξ, η^i and f and leads via (9) to the constants of the motion Φ.

When dealing with dynamical transformations, on the other hand, the analogous procedure does not work since one cannot in general isolate powers of the velocity within the condition (10), due to ξ and η^i being themselves velocity dependent in an unspecified way. This difficulty is, however, circumvented if one assumes a particular form of velocity dependence for these functions. We shall here take the rather simple linear dependence

$$\xi = 0 \tag{11a}$$

$$\eta^i = \chi^i{}_j v^j. \tag{11b}$$

Here $\chi^i{}_j = \chi^i{}_j(t, x)$ has to be a mixed second-rank tensor for η^i to be a vector. In contrast to the n components of η^i in point transformations, it has n^2 components to be extracted via the Noether criterion (10) and so clearly a considerable indeterminary still remains. The resulting conditions for no constraint here take the form (on eliminating $\dot{v}^i$ whenever it occurs with the aid of (7))

$$\chi_{(ij;k)} = 0 \tag{12a}$$

$$\chi_{(ij),t} = 0 \tag{12b}$$

$$2\chi_{(ij)} g^{jk} V_{,k} + f_{,i} = 0 \tag{12c}$$

$$f_{,t} = 0. \tag{12d}$$

Viewing these conditions (12), it appears appropriate to restrict χ_{ij} to be a symmetric tensor. In particular, the following two specialisations prove useful:

$$\text{(i)} \quad \chi_{ij} = \sigma g_{ij} \tag{13}$$

with $\sigma = \sigma(t, x)$ a scalar;

$$\text{(ii)} \quad \chi_{ij} = a\mu_{(i}\lambda_{j)} + b(\mu \cdot \lambda)g_{ij} \tag{14}$$

with $\lambda^i = \lambda^i(t, x)$, $\mu^i = \mu^i(t, x)$ vectors, a and b free constants and the usual dot product defined as

$$\lambda \cdot \mu \equiv \lambda^k \mu_k.$$

Taking the first form (13) for χ_{ij} and substituting into (12), we get immediately

$$\sigma = \text{constant}$$

while a non-trivial symmetry exists (i.e. $\sigma \neq 0$) only if the potential V has the form

$$V(t, x) = V_0(x) + h(t)$$

in which case we get

$$f = -2\sigma V_0$$

with the consequent Noether invariant

$$\Phi = -2\sigma(g_{ij}v^i v^j + V_0(x)). \tag{15}$$

For mechanical problems with $h(t) = 0$, (15) is just the familiar result of energy conservation and obtained by a considerably different procedure in textbooks of mechanics (e.g. Landau and Lifshitz 1960).

The form (13) of χ_{ij} with the single function σ is restrictive enough for the Noether criteria (12) to determine completely this one unknown as a constant and so supply the corresponding single Noether invariant Φ of (15). However, the situation is different for the second assumed form (14) for χ_{ij} where the same criteria are no longer sufficient to fix all the unknowns. For this reason we make a number of further simplifications. Firstly, in order to satisfy (12b) we insist that μ^i and λ^i do not involve the time parameter t, i.e. $\mu^i{}_{,t} = \lambda^i{}_{,t} = 0$. Secondly, we restrict discussion to mechanical situations with a flat manifold and using coordinates in which $g_{ij} = \text{diag}(1, 1, \ldots, 1)$. As a result we may in the subsequent equations write *all* indices (including coordinate labels) as lower indices, since contra and covariant are no longer distinguished. Thirdly, we specify one of the two fields, $\mu^i(x)$ say, to have arbitrary constant components in these coordinates. One is then left with the two equations, subject to the restriction that $f = f(x)$ as required by (12d), namely,

$$a[\mu_{(i}\lambda_{j),k} + \mu_{(k}\lambda_{i),j} + \mu_{(j}\lambda_{k),i}] + b[g_{ij}(\mu \cdot \lambda)_{,k} + g_{ki}(\mu \cdot \lambda)_{,j} + g_{jk}(\mu \cdot \lambda)_{,i}] = 0 \tag{16}$$

$$2[a\mu_{(i}\lambda_{j)}g_{jk} + b(\mu \cdot \lambda)\delta_{ik}]V_{,k} + f_{,i} = 0. \tag{17}$$

The first equation may be solved for λ_i using the arbitrariness of the μ_i. Thus contracting with g_{ij} and equating the coefficients of the μ_i to zero gives

$$[a + (n+2)b]\lambda_{i,k} + a\lambda_{k,i} + a\lambda_{l,l}g_{ki} = 0 \tag{18}$$

with n the dimension of the manifold.

An immediate solution is

$$\left.\begin{array}{ll} \lambda_i = \beta_i & \beta_i = \text{constant} \\ a, b \text{ arbitrary.} & \end{array}\right\} \tag{19}$$

A further possibility arises by contracting (18) with g_{ik} to yield

$$(n+2)(a+b)\lambda_{l,l}=0$$

and setting $a+b=0$ (instead of $\lambda_{l,l}=0$), thereby leading to the equation

$$-(n+1)\lambda_{i,k}+\lambda_{k,i}+\lambda_{l,l}g_{ki}=0. \tag{20}$$

This has beside (19) above, the non-constant solution

$$\left.\begin{aligned} &\lambda_i=\alpha x_i+\beta_i \qquad \alpha, \beta_i = \text{constants} \\ &a+b=0. \end{aligned}\right\} \tag{21}$$

3. Application to physical problems

In this section we apply the above results to the examples of the isotropic harmonic oscillator and the classical Kepler problem. In both cases the Noether point group is smaller than that associated with the equations of motion and so it is of direct interest to see that our extension to velocity dependent symmetries of the very simplest linear type is able to generate the extra constants of the motion in a straightforward way.

3.1. Isotropic oscillator

Here we have

$$V=\tfrac{1}{2}\omega^2(x_1^2+x_2^2+\ldots+x_n^2).$$

Substitution into (21) yields the equations

$$f_{,i}=-2\omega^2[a\mu_{(i}\lambda_{k)}x_k+b(\mu\cdot\lambda)x_i].$$

Of the two possibilities for a, b and λ_i, only (19) yields a non-trivial solution for symmetry, as can be readily seen upon substitution. For this non-trivial case we further set $b=0$, since the possibility $a=0$, $b\neq 0$ just leads to energy conservation as already covered by (15). Now setting $a=1$, without loss of generality we have

$$f_{,i}=-2\omega^2\mu_{(i}\beta_{k)}x_k$$

which has the solution (apart from an additive constant that may be dropped)

$$f=-\omega^2\mu_{(i}\beta_{k)}x^ix^k$$

and gives the Noether constant

$$\Phi=-\mu_{(i}\beta_{k)}(v^iv^k+\omega^2x^ix^k). \tag{22}$$

The arbitrariness of the constants μ_i, β_i in turn demands the separate conservation of the $n(n+1)/2$ quantities $(v^iv^k+\omega^2x^ix^k)$. They do *not* arise as Noether point constants and were originally written down by Fradkin (1965) from more or less intuitive arguments. Since then they were rederived via the Lie method of direct invariance of equations of motion (Leach 1981) and the present dynamical Noether approach complements the above methods.

3.2. The Kepler problem

Here for $n = 3$ we have

$$V = -K/r \qquad r^2 \equiv x_1^2 + x_2^2 + x_3^2.$$

Equation (17) becomes

$$f_{,i} = -(2K/r^3)[a\mu_{(i}\lambda_{k)}x_k + b(\mu \cdot \lambda)x_i].$$

The first type of solution (19) for λ_i allows as the only non-trivial possibility the choice $a = 0$, $b \neq 0$ which again just corresponds to energy conservation as in (15). The second type of solution (21) gives the equations

$$f_{,i} = -(Ka/r^3)\{\alpha[\mu_i r^2 - (\mu \cdot x)x_i] + (\beta \cdot x)\mu_i + (\mu \cdot x)\beta_i - 2(\mu \cdot \beta)x_i\}$$

which, provided all β_i are set to zero, leads to a non-trivial symmetry with

$$f = -Ka\alpha(\mu \cdot x)/r.$$

Additive constants in f are disregarded. New setting $a\alpha = 1$, without loss of generality, we obtain the Noether invariant

$$\Phi = \mu_i[(v \cdot v)x_i - (x \cdot v)v_i - Kx_i/r]. \tag{23}$$

The three separate coefficients of the arbitrary μ_i are just the conserved components of the classical Runge–Lenz vector

$$\boldsymbol{R} \equiv \boldsymbol{v} \times (\boldsymbol{x} \times \boldsymbol{v}) - K\boldsymbol{x}/r.$$

This again can be derived using a point transformation with the aid of the direct Lie method (see Prince and Eliezer 1981) but *not* as a point Noether invariant.

4. Concluding remarks

We have shown that within reasonable limitations it is possible to place dynamical Noether transformations on a systematic footing and thereby considerably extend the usefulness of Noether's theorem. In fact the theorem has already been applied in the past by Levy-Leblond (1971) to linear dynamical transformations but written down only in an *ad hoc* manner for a number of problems (including those treated here), and it is hoped that our present contribution will add a suitable basis to this otherwise instructive work.

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